"Guys, don't worry about midterms. They're not the best measure of your worth as a physicist. In fact, I did rather poorly on my first physics midterm, I got something like 3 out of 40. Of course, everyone else got 1 out of 40, but that's not really the point..."

- Prof. Nima Arkani-Hamed, UC Berkeley

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

Boundary conditions at the interface ($y = 0$) between Region 1 and Region 2 imply there will be EM waves propagating in the $\hat{x}$-direction in both regions. Without knowing their polarizations just yet, let the electric fields of the waves be given by:

$$\begin{aligned} &\mathbf{E}_1 e^{i(k_1 x - \omega t)} \\ &\mathbf{E}_2 e^{i(k_2 x - \omega t)}. \end{aligned} \tag{1}$$

At the interface between Region 1 and Region 2, since the electric and magnetic field amplitudes in both regions are independent of x and t, the only way to satisfy boundary conditions at all positions and times is if:

$$e^{i(k_1 x - \omega t)} = e^{i(k_2 x - \omega t)} \tag{2}$$

For example, at $t = 0$, this requires that:

$$k_1 = k_2 \tag{3}$$

which means the indices of refraction in the two regions must be the same. Thus we have,

$$\sqrt{\frac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0}} = \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_0 \mu_0}} \tag{4}$$

which gives us $\mu_2 = 4\mu_0$.

(b)

We have the following boundary conditions at the interface:

$$\begin{aligned} \epsilon_1 E_\perp^{(1)} &= \epsilon_2 E_\perp^{(2)} \\ \mu_1 H_\perp^{(1)} &= \mu_2 H_\perp^{(2)} \\ E_\parallel^{(1)} &= E_\parallel^{(2)} \\ H_\parallel^{(1)} &= H_\parallel^{(2)} \end{aligned} \tag{5}$$

First let's see if $\vec{E}$ can be along $\hat{y}$. In this case, from Eqs. (5), we have that:

$$\begin{aligned} \epsilon_1 E_y^{(1)} &= \epsilon_2 E_y^{(2)} \Rightarrow \\ 4E_y^{(1)} &= E_y^{(2)} \end{aligned} \tag{6}$$

and we also know that for the wave in Region 1:

$$\begin{aligned} H_z^{(1)} &= \sqrt{\frac{\epsilon_1}{\mu_1}} E_y^{(1)} \Rightarrow \\ H_z^{(1)} &= 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(1)}, \end{aligned} \tag{7}$$

and for the wave in Region 2:

$$\begin{aligned} H_z^{(2)} &= \sqrt{\frac{\epsilon_2}{\mu_2}} E_y^{(2)} \Rightarrow \\ H_z^{(2)} &= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(2)} = 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(1)} \end{aligned} \tag{8}$$

We see that these relations verify that $H_z^{(1)} = H_z^{(2)}$ as demanded by the boundary conditions in Eqs. (5). The relations are consistent, so therefore such a polarization is allowed.

Can $\vec{E}$ be along $\hat{z}$? We have from Eqs. (5) that

$$E_z^{(1)} = E_z^{(2)} \tag{9}$$

For the wave in Region 1:

$$\begin{aligned} H_y^{(1)} &= \sqrt{\frac{\epsilon_1}{\mu_1}} E_y^{(1)} \Rightarrow \\ H_y^{(1)} &= 2\sqrt{\frac{\epsilon_0}{\mu_0}} E_z^{(1)}. \end{aligned} \tag{10}$$

In Region 2:

$$H_y^{(2)} = \sqrt{\frac{\epsilon_2}{\mu_2}} E_y^{(2)} \Rightarrow$$
$$H_y^{(2)} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_y^{(1)}. \tag{11}$$

We see that this verifies the boundary condition on H_y from Eqs. (5):

$$\mu_1 H_y^{(1)} = 2\sqrt{\epsilon_0\mu_0}E_y^{(1)} =$$
$$\mu_2 H_y^{(2)} = 2\sqrt{\epsilon_0\mu_0}E_z^{(1)}. \tag{12}$$

So in fact the wave can be in either polarization state.

Problem 2

Reflection at the Brewster angle transmits all TM light and reflects part of the TE light. The degree of polarization P (for linear polarized light) is given by Fowles Eq. (2.27):

$$P = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{13}$$

The light is initially unpolarized, so it is 50% TE and 50% TM. The light transmitted has reduced TE intensity (since some is reflected at the interface), but the TM intensity remains the same. Consequently, the transmitted light is partially polarized:

$$P = \frac{I_{TM} - I_{TE''}}{I_{TM} + I_{TE''}} = \frac{I_{TE'}}{I_{TM} + I_{TE''}}. \tag{14}$$

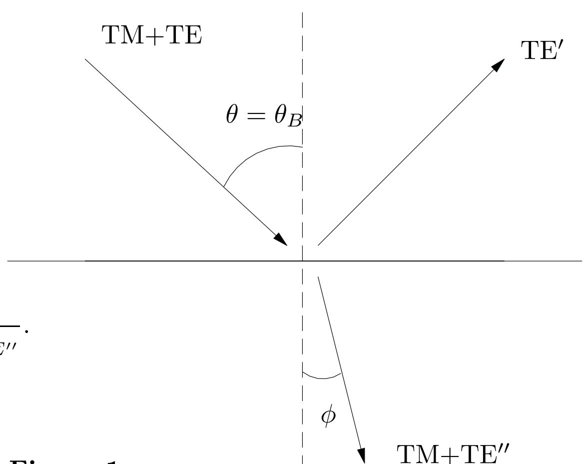

Figure 1

Now we can apply the Fowles formalism for reflection and transmission of TM and TE waves, using the definitions on page 43 of Fowles:

$$r_s = \left[\frac{E'}{E}\right]_{TE}$$
$$t_s = \left[\frac{E''}{E}\right]_{TE}$$
$$r_p = \left[\frac{E'}{E}\right]_{TM}$$
$$t_p = \left[\frac{E''}{E}\right]_{TM} \tag{15}$$

From the boundary conditions for **E** and **H** at the interface (Eqs. (5)), we find some basic relations between the quantities in Eqs. (15):

$$t_s = r_s + 1$$
$$nt_p + r_p = 1 \tag{16}$$

At the Brewster angle $r_p = 0$, so $t_p = 1/n$. Employing the fact that $I \propto nE^2$ (thanks to Paul Wright!), we have for the partial polarization:

$$P = \frac{1}{n}\frac{r_s^2}{t_s^2 + t_p^2} = \frac{1}{n}\frac{(t_s-1)^2}{t_s^2 + 1/n^2} \tag{17}$$

Fowles works out the general formulae (Fresnel's equations) for reflection/refraction at a plane interface, and in particular for t_s we have from Fowles (2.56):

$$t_s = \frac{2\cos\theta\sin\phi}{\sin(\theta+\phi)}. \tag{18}$$

If we combine Fowles Eq. (2.64)

$$\tan\theta_B = n \tag{19}$$

with Snell's Law

$$n = \frac{\sin\theta}{\sin\phi}, \tag{20}$$

we have the Brewster condition:

$$\theta + \phi = \pi/2, \tag{21}$$

which follows from the fact that $\sin\phi = \cos\theta$. Knowing also then from Eq. (21) that $\sin(\theta+\phi) = 1$, we find that:

$$t_s = 2\cos^2\theta \tag{22}$$

Using the trigonometric identity $\sec^2 = \tan^2 + 1$ and Eq. (19) we find that $\cos^2\theta = 1/(n^2+1)$ so

$$t_s = \frac{2}{n^2+1} \tag{23}$$

Plugging Eq. (23) into Eq. (17), with a little algebra, gives us:

$$P = \frac{n[(n^2-1)]^2}{1+6n^2+n^4}. \tag{24}$$

For glass, where $n = 1.5$, we have that $P \approx 12\%$.

Problem 3

(a)

Note that the Brewster condition (Eq. (21)) is met. Thus from Eq. (19) we have that $n = \tan 60^o = \sqrt{3}$. Since $\mu = \mu_0$, we find that $\epsilon = 3\epsilon_0$.

(b)

Since the Brewster condition is met, the reflected light is 100% TE. Therefore reflected light is linearly polarized along $\hat{y}$.

(c)

The light is initially circularly polarized, so it is an equal superposition of linear polarizations (TE and TM). Since the TM component has 100% transmission, it suffices to consider the transverse electric case ($\vec{E}$ along $\hat{y}$) where all reflection occurs. We have the boundary condition that $E_\parallel$ is continuous, so

$$E_i + E_r = E_t. \tag{25}$$

Also $H_\parallel$ is continuous, and we have for the incident, reflected and transmitted waves the following components of H in the $\hat{x}$ direction:

$$H_x^{(i)} = -E_i\sqrt{\frac{\epsilon_0}{\mu_0}}\cos 60^o$$
$$H_x^{(r)} = E_r\sqrt{\frac{\epsilon_0}{\mu_0}}\cos 60^o$$
$$H_x^{(t)} = -E_t\sqrt{\frac{3\epsilon_0}{\mu_0}}\cos 30^o \tag{26}$$

From which we have the condition:

$$-E_i + E_r = -3E_t. \tag{27}$$

Adding Eq. (25) to Eq. (27) gives us $(E_r/E_i = 1/2)_{TE}$, or $(I_r/I_i = 1/4)_{TE}$. The intensity in the TE component is half the initial intensity, so in total $I_r/I_i = 1/8$.

Problem 4

Here we treat the tungsten filament as a relatively long straight wire of thickness $s = 0.1$ mm. The distance between the filament and an aperture is r. We want a transverse coherence width $l_t \geq 1$ mm. Then from Fowles Section 3.7, and in particular Eq. (3.42), we find that:

$$l_t = \frac{r\lambda}{s} \geq 1mm \tag{28}$$

If we assume that the tungsten lamp has a central wavelength of 5000 $\mathring{A}$, then Eq. (28) demands that $r \geq 200$ mm.

If a double-slit aperture is used, the slits should be oriented parallel to the lamp filament, otherwise the thickness of the wire s would have to be replaced with the length of the wire, which is naturally much greater than s. This would force r to be much greater.

Problem 5

The power spectrum of the Gaussian pulse $f(t)$ is given by $G(\omega) = |g(\omega)|^2$, where in our case $g(\omega)$ is:

$$g(\omega) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt \exp\left(-at^2 + i(\omega-\omega_0)t\right) \tag{29}$$

It is first useful to derive a result about Gaussian integrals. It turns out that $\int_{-\infty}^{\infty} e^{-ax^2}dx$ converges, so let's set it equal to some constant c. Now consider the integral over the entire plane:

$$\int_{-\infty}^{\infty} e^{-ax^2}dx \int_{-\infty}^{\infty} e^{-ay^2}dy = c^2. \tag{30}$$

Next we convert the integral into polar coordinates ($r^2 = x^2 + y^2$).

$$\int_0^{\infty} e^{-ar^2} r dr \int_0^{2\pi} d\phi = c^2. \tag{31}$$

Which is relatively straightforward with the substitution $u = ar^2$, $du = 2ardr$. From this integral we find that $c^2 = \pi/a$.

Thus all we need to do is to convert the integral in Eq. (29) to a form resembling a Gaussian. This can be done by completing the square in the exponent:

$$-at^2 + i(\omega - \omega_0)t = -a\left[\left(t^2 - 2i\frac{\omega - \omega_0}{2a}t - \frac{(\omega-\omega_0)^2}{4a^2}\right) + \frac{(\omega-\omega_0)^2}{4a^2}\right]$$
$$= -a\left(t - i\frac{\omega-\omega_0}{2a}\right)^2 - \frac{(\omega-\omega_0)^2}{4a} \tag{32}$$

Now the integral in Eq. (29) is just a Gaussian integral, which is no longer a problem...

Working through the constants gives us:

$$g(\omega) = \frac{A}{\sqrt{2a}}\exp\left[-\frac{(\omega-\omega_0)^2}{4a}\right]. \tag{33}$$

$G(\omega) = |g(\omega)|^2$ is clearly of the same form, and so $G(\omega)$ is a Gaussian function centered at ω_0.

Problem 6

The condition for a fringe maximum to occur is given by Fowles Eq. (4.10):

$$2N\pi = \frac{4\pi}{\lambda}nd\cos\theta + \delta_r. \tag{34}$$

Use the small angle approximation:

$$2N\pi = \frac{4\pi}{\lambda}nd\left(1 - \frac{\theta^2}{2}\right) + \delta_r. \tag{35}$$

We are told that the zeroth order fringe ($N = 0$) has zero radius ($\therefore \theta = 0$), so we have:

$$\delta_r = -\frac{4\pi}{\lambda}nd \tag{36}$$

and subsequently:

$$2N\pi = \frac{4\pi}{\lambda}nd\frac{\theta^2}{2}. \tag{37}$$

So $\theta \propto \sqrt{N}$. Since the radius of the fringes r is proportional to θ, it follows immediately that $r \propto \sqrt{N}$.

Problem 7

This solution follows directly from the discussion of antireflecting films in Fowles (page 99). We want to choose the thickness of the film to be $\frac{\lambda}{4}$. Then the reflectance is zero if the index of refraction of the coating $n = \sqrt{\frac{\epsilon}{\epsilon_0}}$.

Problem 8

Fowles Eq. (4.24) states:

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} t \tag{38}$$

which is equivalent to:

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} E_0 + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} E_0' = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} E_T \tag{39}$$

The total **E** and **H** just to the right of the right-hand interface are:

$$E_{RH} = E_T$$
$$H_{RH} = n_T E_T. \tag{40}$$

The total **E** and **H** just to the left of the left-hand interface are:

$$E_{LH} = E_0 + E_0'$$
$$H_{LH} = n_0 E_0 - n_0 E_0'. \tag{41}$$

When the relations in Eqs. (40) and (41) are substituted into Eq. (39), we find that:

$$\begin{bmatrix} E_{LH} \\ H_{LH} \end{bmatrix} = M \begin{bmatrix} E_{RH} \\ H_{RH} \end{bmatrix} \tag{42}$$

Therefore the overall transfer matrix M_{tot} is merely the product of transfer matrices for the individual films M_i. This follows from induction. Suppose there are n films with transfer matrices $M_1, M_2, ..., M_n$. Let the fields to the right of the last film be E_n and H_n. The fields just to the left of the nth film are:

$$\begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix} = M_n \begin{bmatrix} E_n \\ H_n \end{bmatrix} \tag{43}$$

the fields just to the left of the (n-1)th film are:

$$\begin{bmatrix} E_{n-2} \\ H_{n-2} \end{bmatrix} = M_{n-1} \begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix} = M_{n-1} M_n \begin{bmatrix} E_n \\ H_n \end{bmatrix} \tag{44}$$

and so on...

This argument leads to Fowles Eq. (4.28) as stated.